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MASS AND HEAT TRANSFER TO SPHERES, CYLINDERS AND PLANAR SURFACES: A UNIFIED "FILM" MODEL DESCRIPTION

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ABSTRACT

We develop a unified film model description of mass and heat transfer from binary mixtures to spherical, cylindrical and planar surfaces. The model shows that while the Sherwood (Nusselt) number has a lower asymptotic value of 2, there is no corresponding asymptote for transfer to cylindrical objects. The influence of finite mass transfer fluxes on the mass and heat transfer coefficients is described in a unified manner for all geometries in terms of a generalized Ackermann factor. This correction factor is independent of geometry. **© 2001 Elsevier Science Ltd**

Introduction

In several applications of interest to the processing industries we encounter transfer of heat and mass to planar, cylindrical or spherical surfaces. In coal combustion and gasification, for example, we encounter simultaneous heat and mass transfer between spherical coal particles with a gaseous mixture. During condensation we encounter heat and mass transfer to cylindrical tubes. Evaporation of water from spills on oceans involves mass transfer from (nearly) planar surfaces. Often in the process industries the fluid flows across, or along, the surface in turbulent flow. In the "film" model for heat and mass transfer, the transfer resistance is assumed to be located in a thin layer adjacent to the surface. Transfer within this film is by molecular transport. In Fig. 1, the effective "film" is located between $r = r_0$ and $r = r_\delta$. Some authors [1,2,3] ignore the curvature of the film in the case of cylindrical and spherical surfaces and take $(r_\delta - r_0)$ as the value of the film thickness to calculate the heat and mass transfer coefficients. This definition of the film thickness is correct only for planar surfaces and its incorrect application to spherical films, led Kalson [2] to erroneously conclude that the Ackermann correction factor [4] for finite mass transfer rates is affected by the system geometry.

There is also another issue which needs clarification. For mass transfer to spherical surfaces, the Sherwood (or Nusselt) number has a lower asymptotic value of 2. For transfer to cylindrical surfaces, on

the other hand, measurements at low Reynolds numbers do not appear to show any lower limit [6]. The question arises as to the compatibility of the results for transfer to spheres and cylinders.

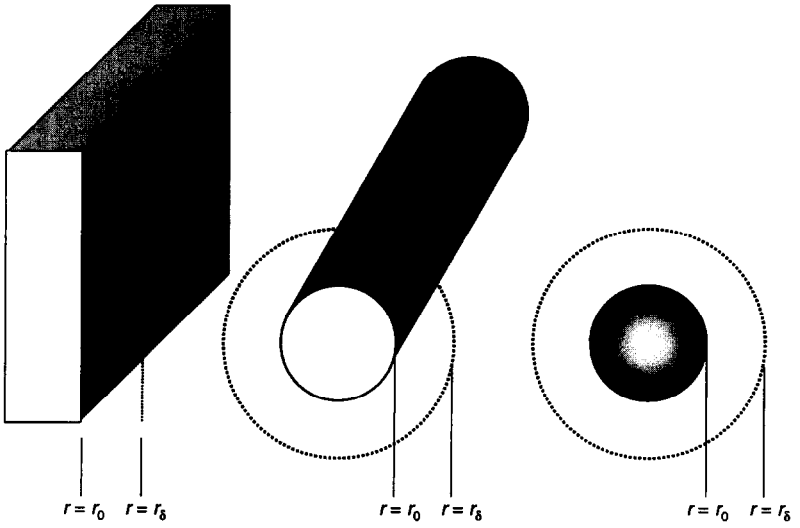

FIG. 1
Film thicknesses in planar, cylindrical and spherical geometries.

The purpose of the present communication is to develop a unified film model description of mass transfer valid for spheres, cylinders and planar surfaces. We start by considering mass transfer in binary mixtures and subsequently generalise our treatment to simultaneous transfer of heat and mass. We develop the proper definitions of the "film" thickness and the Sherwood and Nusselt numbers. The issue of the lower asymptotic value of Nu and Sh numbers for spheres and cylinders is then discussed.

Film Model for Mass Transfer in Binary Mixtures

Consider steady state mass transfer, to planar, cylindrical and spherical surfaces, in an isothermal binary fluid mixture made up of components 1 and 2. The mass transfer is assumed to be described by molecular diffusion within the planar, cylindrical and spherical regions defined within the limits $r = r_0$ and $r = r_\delta$; see Fig. 1.

The equation of continuity of moles of species can be rewritten for each individual species

$$\frac{d\left(r^\alpha N_i\right)}{dr} = 0; \quad i = 1,2 \tag{1}$$

and for the total mixture

$$\frac{d\left(r^\alpha N_t\right)}{dr} = 0; \quad N_t = N_1 + N_2 \tag{2}$$

The exponent α in Eqs (1) and (2) depends on geometry:

$\alpha = 0$ for planar film; $\alpha = 1$ for cylindrical film; $\alpha = 2$ for spherical film (3)

Equations (1) and (2) show that $r^{\alpha} N_i$ is r-invariant, i.e.

$$r_0^{\alpha} N_{i0} = r_{\delta}^{\alpha} N_{i\delta} = r^{\alpha} N_i \tag{4}$$

In the following discussions we evaluate the fluxes N_i at the surface, i.e at $r = r_0$ and denote these by N_{i0}.

In order to calculate the fluxes N_{i0} we need a constitutive equation and the Maxwell-Stefan relations are the best candidates for this purpose [7,8,9]:

$$\frac{dx_1}{dr} = \frac{x_1 N_{20} - x_2 N_{10}}{c_t D_{12}} \tag{5}$$

where D_{12} is the Fick diffusivity for the 1-2 binary fluid mixture. The fluxes N_{i0} are with respect to a stationary coordinate reference frame and are made up of two contributions: a diffusive flux and a "drift" contribution:

$$N_{i0} = J_{i0} + x_{i0} N_{i0}; \quad i = 1,2 \tag{6}$$

The diffusive flux is given by

$$J_{i0} = -c_t D_{12} \left. \frac{dx_i}{dr} \right|_{r=r_0} \quad ; \quad i = 1,2 \tag{7}$$

Substituting $x_2 = 1 - x_1$ and re-arranging we obtain

$$\frac{dx_1}{dr} = \frac{x_1 (N_{10} + N_{20})}{c_t D_{12}} - \frac{N_{10}}{c_t D_{12}} \tag{8}$$

It is convenient to define the following variables:

(a) a generalized film thickness ℓ :

$$\begin{aligned} &\ell = r_{\delta} - r_0 \text{ for planar films} \\ &\ell = r_0 \ln(r_{\delta}/r_0) \text{ for cylindrical films;} \\ &\ell = r_0 (1 - r_0/r_{\delta}) \text{ for spherical films;} \end{aligned} \tag{9}$$

(b) a dimensionless distance coordinate η

$$\begin{aligned} &\eta = \frac{r - r_0}{r_{\delta} - r_0}; \quad dr = \ell d\eta; \text{ for planar films} \\ &\eta = \frac{\ln(r/r_0)}{\ln(r_{\delta}/r_0)}; \quad dr = \left(\frac{r}{r_0}\right) \ell d\eta; \text{ for cylindrical films;} \\ &\eta = \frac{(1/r - 1/r_0)}{(1/r_{\delta} - 1/r_0)}; \quad dr = \left(\frac{r}{r_0}\right)^2 \ell d\eta; \text{ for spherical films;} \end{aligned} \tag{10}$$

(c) dimensionless mass transfer rate factors for mixture, ϕ_t , and for component 1, ϕ_1

$$\phi_t = \frac{N_{10} + N_{20}}{c_t D_{12}/\ell}; \quad \phi_1 = \frac{N_{10}}{c_t D_{12}/\ell} \tag{11}$$

With these definitions the Maxwell-Stefan diffusion equations simplify to the following dimensionless form

$$\frac{dx_1}{d\eta} = \phi_t x_1 - \phi_1 \tag{12}$$

which is to be solved with the boundary conditions

$$\begin{aligned} r = r_0; \quad \eta = 0; \quad x_1 = x_{10} \\ r = r_\delta; \quad \eta = 1; \quad x_1 = x_{1\delta} \end{aligned} \tag{13}$$

The solution to the linear differential eq. (12) subject to the boundary conditions (11) is

$$\frac{x_1 - x_{10}}{x_{1\delta} - x_{10}} = \frac{\exp(\phi_t \eta) - 1}{\exp(\phi_t) - 1} \tag{14}$$

The diffusive flux J_{10} can be calculated from eq. (7); the result is

$$J_{10} = -\frac{c_t D_{12}}{\ell} \left.\frac{dx_{10}}{d\eta}\right|_{\eta=0} = c_t \, k \, \Xi (x_{10} - x_{1\delta}); \quad k \equiv \frac{D_{12}}{\ell}; \quad \Xi \equiv \frac{\phi_t}{\exp(\phi_t) - 1} \tag{15}$$

where we have defined the mass transfer coefficient k as the Fick diffusivity divided by the generalized film thickness ℓ. Some authors in the literature [1,2,3] have taken this film thickness to be equal to $(r_\delta - r_0)$; this definition is erroneous and does not take the curvature of the film into account.

The factor Ξ, defined in eq. (15), corrects the transfer fluxes for finite mass transfer rates. This factor was derived independently by Ackermann [4] and Colburn and Drew [5]. A plot of Ξ vs ϕ_t, given in Fig. 2, shows that this correction could be significant for high net mixture fluxes. Equation (15) only allows calculation of the diffusive contribution to the fluxes N_{i0} and an additional equation required to calculate the two fluxes N_{10} and N_{20}. This additional relation has been termed the bootstrap relation [8,9] and, in general, takes the form of a linear constraint on the fluxes

$$\Lambda_1 N_{10} + \Lambda_2 N_{20} = 0; \quad N_{20} = -\left(\frac{\Lambda_1}{\Lambda_2}\right) N_{10} \tag{16}$$

This additional bootstrap relations allow us to determine the total flux N_{t0} from

$$\phi_t \equiv \frac{N_{10} + N_{20}}{c_t D_{12}/\ell} = \ln\left(\frac{1 - x_{10}\dfrac{\Lambda_1 - \Lambda_2}{x_{10}\Lambda_1 + x_{20}\Lambda_2}}{1 - x_{1\delta}\dfrac{\Lambda_1 - \Lambda_2}{x_{1\delta}\Lambda_1 + x_{2\delta}\Lambda_2}}\right) \tag{17}$$

Equations (16) and (17) covers most cases of importance to chemical and mechanical engineers, which are listed below.

(a) equimolar diffusion

$$N_{10} + N_{20} = 0; \quad \Lambda_1 = \Lambda_2; \quad \phi_t = 0; \quad \Xi \equiv \frac{\phi_t}{\exp(\phi_t) - 1} = 1 \tag{18a}$$

(b) Stefan diffusion

$$N_{20}=0;\quad \Lambda_2\neq 0;\Lambda_1=0;\quad \phi_t=\ln\left(\frac{x_{2\delta}}{x_{20}}\right)=\ln\left(\frac{1-x_{1\delta}}{1-x_{10}}\right) \tag{18b}$$

(c) Graham diffusion inside porous media

$$\Lambda_i=\sqrt{M_i}\ \text{where } M_i \text{ is the molar mass} \tag{18c}$$

(d) non-equimolar distillation with different molar enthalpies of vaporizations of individual species

$$\Lambda_i=H_i^v-H_i^x \tag{18d}$$

(e) diffusion with heterogeneous chemical reactions where the flux ratios are fixed by reaction stoichiometry

$$\nu_1 A_1+\nu_2 A_2=0;\quad N_{20}=-\left(\frac{\nu_2}{\nu_1}\right)N_{10};\quad \left(\frac{\Lambda_1}{\Lambda_2}\right)=-\left(\frac{\nu_2}{\nu_1}\right); \tag{18e}$$

(f) condensation of a vapour mixture to form a liquid condensate with specified mole fraction z_i

$$\frac{N_{10}}{N_{20}}=\left(\frac{z_1}{z_2}\right);\quad \left(\frac{\Lambda_1}{\Lambda_2}\right)=-\left(\frac{z_2}{z_1}\right) \tag{18f}$$

(g) Evaporation of a liquid mixture to form a vapour with specified mole fraction z_i

$$\frac{N_{10}}{N_{20}}=\left(\frac{z_1}{z_2}\right);\quad \left(\frac{\Lambda_1}{\Lambda_2}\right)=-\left(\frac{z_2}{z_1}\right) \tag{18g}$$

Equation (15), in combination with the bootstrap relation (16), allows calculation of the two fluxes N_{10} and N_{20}.

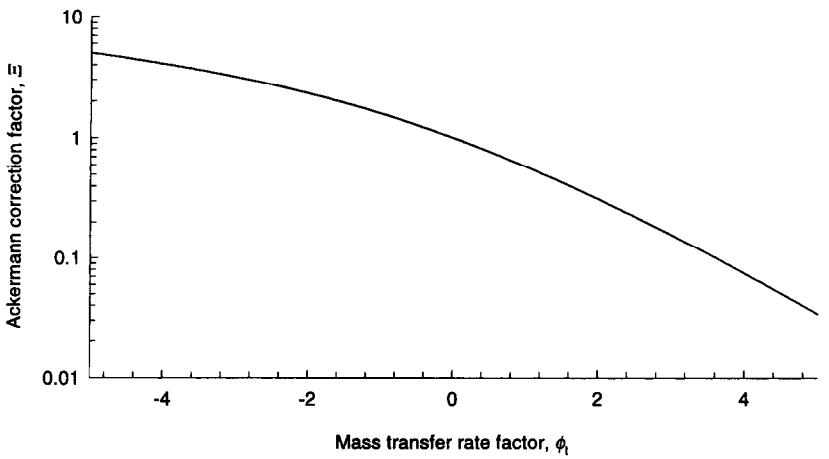

FIG. 2
Ackermann correction factor as a function of the mass transfer rate factor.

Simultaneous Heat and Mass Transfer

For simultaneous heat and mass transfer in binary fluid mixtures, we have to contend additionally with the equations of continuity of energy:

$$\frac{d\left(r^{\alpha}E\right)}{dr}=0 \tag{19}$$

where E, the energy flux, is made up of the conductive and convective enthalpy contributions:

$$E=q+(N_1H_1+N_2H_2);\quad q=-\lambda\frac{dT}{dr};\quad E=q+(N_1C_{p1}+N_2C_{p2})(T-T_{ref}) \tag{20}$$

where λ is the thermal conductivity of the flux mixture. Defining the heat transfer coefficient and the heat transfer rate factor as follows

$$h\equiv\frac{\lambda}{\ell};\quad \phi_H\equiv\frac{(N_1C_{p1}+N_2C_{p2})}{h} \tag{21}$$

we rewrite eq. (20) in the convenient form

$$E=-h\frac{dT}{d\eta}+h(T-T_{ref}) \tag{22}$$

Equation (22) can be solved for the boundary conditions

$$\begin{aligned} &r=r_0;\quad \eta=0;\quad T=T_0\\ &r=r_\delta;\quad \eta=1;\quad T=T_\delta \end{aligned} \tag{23}$$

to obtain the temperature profiles

$$\frac{T-T_0}{T_\delta-T_0}=\frac{\exp(\phi_H\eta)-1}{\exp(\phi_H)-1} \tag{24}$$

whence we obtain the conductive heat flux

$$q_0=-h\left.\frac{dT}{d\eta}\right|_{\eta=0}=h\frac{\phi_H}{\exp(\phi_H)-1}(T_0-T_\delta)=h\,\Xi_H(T_0-T_\delta) \tag{25}$$

Equation (25) is the heat transfer analog of eq. (15). The high-flux correction factor Ξ_H, in common with its mass transfer analog, Ξ, is independent of the geometry. This is in contrast with the conclusion reached by Kalson [2] who concluded that the Ackermann correction factor is geometry dependent; the error in this paper stems from an incorrect definition of the film thickness for the spherical geometry as being equal to $(r_\delta-r_0)$.

For mass and heat transfer to cylindrical and spherical objects it is conventional to define the Sherwood and Nusselt numbers as $Sh\equiv\frac{k(2r_0)}{D_{12}}$; $Nu\equiv\frac{h(2r_0)}{\lambda}$ and therefore we obtain the following result

$$Sh = Nu = \frac{2}{\ln(r_\delta / r_0)} \quad \text{for cylindrical films;}$$
$$Sh = Nu = \frac{2}{(1 - r_0 / r_\delta)} \quad \text{for spherical films;} \tag{26}$$

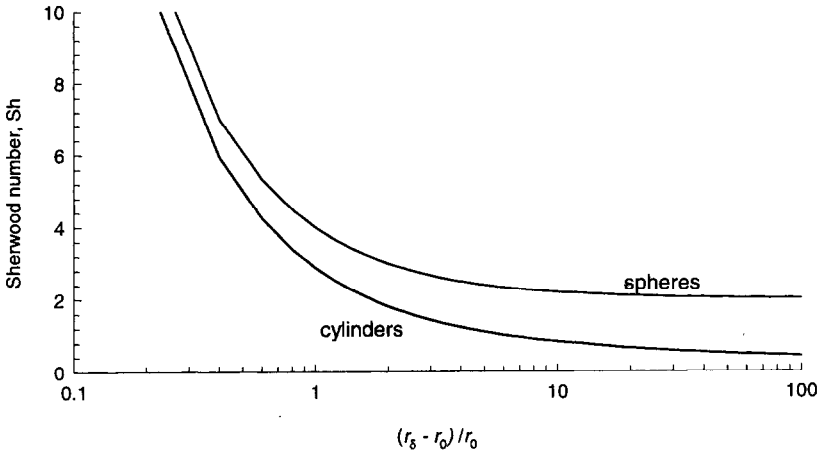

FIG. 3
Sherwood (Nusselt) number as a function of the dimensionless thickness $(r_\delta - r_0)/r_0$.

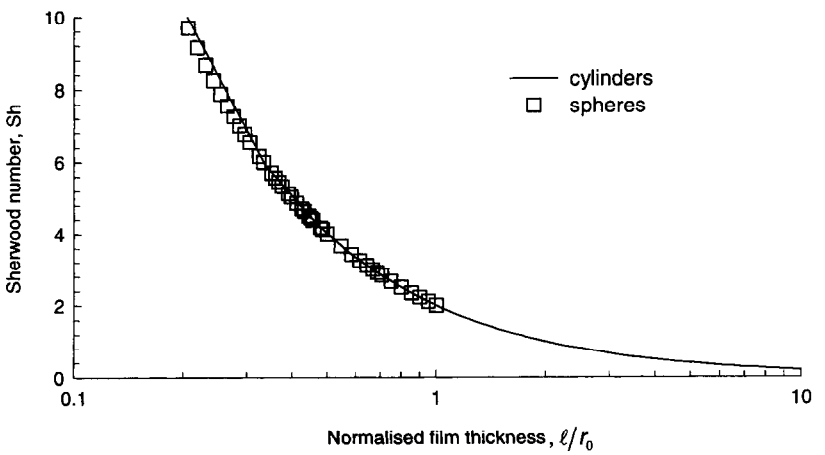

FIG. 4
Sherwood (Nusselt) number as a function of the normalized film thickness ℓ/r_0.

A plot of Sh (=Nu) vs the parameter $(r_\delta - r_0)/r_0$ is given Fig. 3 or cylinders and spheres, respectively. For transfer to spheres the Sh (=Nu) approaches a value the lower asymptotic limiting value of 2. For transfer to cylinders, there is no asymptotic limit. Experimental data on mass transfer to cylinders in laminar crossflow [6] confirm that the Nusselt number values as low as 0.2 can be obtained. When the Sh (or Nu) numbers for transfer to spheres or cylinders are compared at the same value of the normalized film thickness ℓ/r_0, the values are identical; see Fig. 4. The open square symbols in Fig. 4, representing the calculations for spheres, do not extend beyond values of ℓ/r_0 exceeding unity, because the limiting value of $\ell/r_0 = 1$ for "infinitely" thick films. There is no corresponding limiting value of ℓ/r_0 for the cylindrical geometry. The equality of Sh (and Nu) for spheres and cylinders for equal values of ℓ/r_0, below unity, is a new result which is not reported so far in the literature.

Conclusions

We have developed a generalized film model for planar, cylindrical and spherical geometries. If the film thickness is defined in the proper manner, a general method can be developed for calculation of the heat and mass transfer coefficients and fluxes. Our analysis shows that the Sh (Nu) number for mass transfer to spheres has a lower asymptote of 2, but for cylinders there is no corresponding lower asymptote. When Sh (or Nu) numbers for spheres and cylinders are compared at the same normalized film thickness ℓ/r_0, the values are identical.

The Ackermann correction factors for mass and heat transfer, Ξ and Ξ_H, are independent of the geometry. A different conclusion has been reached by Kalson [2] due to an incorrect definition of the film thickness.

Nomenclature

A_1, A_2	denoting the chemical species participating in surface chemical reaction, see eq. (18e)
c_t	total molar concentration of the fluid mixture, mol m^{-3}
C_p	molar heat capacity, J mol^{-1} K^{-1}
D	Fick diffusivity in binary mixture, m^2 s^{-1}
E	energy flux, W m^{-2}
h	heat transfer coefficient, W m^{-2} K^{-1}
H_i	partial molar enthalpy of species i, J mol^{-1}
J_i	molar diffusion flux of species i, mol m^{-2} s^{-1}
k	matrix of multicomponent mass transfer coefficients, m s^{-1}

ℓ generalized definition of film thickness, defined in eq. (9), m

M_i molar mass of species i, kg mol^{-1}

N_i molar flux of species i in laboratory fixed reference frame, mol m^{-2} s^{-1}

N_t mixture molar flux, mol m^{-2} s^{-1}

Nu Nusselt number, dimensionless

q conductive heat flux, W m^{-2}

r radial distance coordinate, m

Sh Sherwood number, dimensionless

T absolute temperature, K

x_i mole fraction of species i, dimensionless

z_i mole fraction of condensate or vaporized liquid, dimensionless

Greek Letters

α =0 for planar surface, = 1 for cylinders, = 2 for spheres, dimensionless

η dimensionless coordinate distance coordinate defined by eq. (10), dimensionless

λ thermal conductivity of fluid mixture, W m^{-1} K^{-1}

Λ coefficients defined in eq. (16), dimensionless

ν stoichiometric coefficient, dimensionless

ϕ mass transfer rate factor, dimensionless

ϕ_H heat transfer rate factor, dimensionless

Ξ Ackermann correction factor, dimensionless

Subscripts

1 referring to component 1

2 referring to component 2

0 at position $r = r_0$

δ at position $r = r_\delta$

H heat transfer parameter

ref reference state

t total mixture

Superscripts

x referring to liquid phase

y referring to vapour phase

References

1. J. Chomiak, In *Combustion, A study in theory, fact and application* (see page 222), Gordon and Breach, New York, 1990.

2. P.H. Kalson, *Ind.Eng.Chem. Fundamentals*, **22**, 355 (1983).

3. J. Tomeczek, *Coal Combustion* (see p. 34), Malabar, Florida, Krieger, 1994.

4. G. Ackermann, *Verein Deutscher Ingenieure Forschungsheft*, **382**, 1 (1937).

5. A.P. Colburn and T.B. Drew, *Transactions of the AIChE*, **33**, 197 (1937).

6. C.M. Lange, F. Durst and M. Breuer, *Int.J. Heat & Mass Transfer*, **41**, 3409 (1998).

7. R. Krishna and J.A. Wesselingh, *Chem. Eng. Sci.*, **52**, 861 (1997).

8. J.A. Wesselingh and R. Krishna, *Mass Transfer in Multicomponent Mixtures*, Delft University Press, Delft, 2000.

9. R. Taylor and R. Krishna, *Multicomponent mass transfer*, John Wiley, New York, 1993.

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